Non-modal Analysis of Buoyancy-Driven Instabilities in Porous Media of a Two-Layer Miscible Stratification in the Presence of Differential Diffusion

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Abstract In a porous medium, a two-layer miscible stratification in the presence of differential diffusion is subject to buoyancy-briven instabilities for certain values of the parameters. For such systems, drawing reliable information from linear stability analysis is complex as the underlying base states are time evolving and the linearized operators are also non-normal. Here, we analyze the stability problem through the non-modal approach that takes these two features into account. For the delayed-double diffusive instability, it is shown that the non-modal analysis predictions are significantly different from those of the linear stability analysis based on the quasi-steady-state approximation. This is shown by considering the maximum amplification that the system can undergo and the wavenumber of the optimal perturbations.

1 Introduction

Porous media flows are encountered in many applications including geophysical flows (for example CO_2 sequestration), industrial processes or biological flows. We analyze here the stability of a horizontal interface between two miscible solutions in the gravity field using linear stability non-modal techniques. Depending on the parameters, several mechanisms of buoyancy-driven instabilities can be present [8]. For example, the Rayleigh–Taylor (RT) instability occurs when a denser solution overlies a less dense one, while the double diffusive (DD) instability is present when the lower solution is denser than the upper one, but contains a solute that diffuses faster than the one of the upper solution [6]. Other types of instabilities include the diffusive-layer convection instability (DLC), as well as the delayed-double diffusive instability (DDD), both of which are stable at initial time but develop later on because of the evolving density profile.

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From the point of view of linear stability analysis (LSA), the problem is characterised by a time-dependent non-normal operator resulting from the linearisation of the governing equations, which complicates the analysis for two reasons. First, the traditional LSA requires the eigendecomposition of the linear operator but non-normality implies the non-orthogonality of its eigenvectors. As a result, a combination of eigenvectors can undergo transient growth even if all individual eigenvectors are decaying and this can lead to a bypass transition to the non-linear regime [4]. Second, as the base state of the linearised equations is time-dependent, a one-time eigendecomposition based on a quasi-steady-state approximation (QSSA) may not be appropriate. Indeed, this analysis assumes that the base state evolves slowly enough so it can be frozen at an initial time [7], but for the configurations considered here the evolution of the base state can be quite fast in the initial stages of evolution.

We therefore study the problem using time dependent non-modal analysis (NMA) in which both difficulties are addressed [4]. This method allows to compute the initial condition and the corresponding wavenumber that is the most amplified (optimal perturbation) at a given time and as a corollary, the initial condition that achieves maximum amplification globally in time (for asymptotically stable systems). In the context of CO_2 sequestration in saline aquifers, the NMA approach has already been considered as this convective dissolution problem also gives rise to a non-normal time-dependent linear operator [1, 3, 5]. The situation here is however quite different as we consider a two-species miscible solution with no imposed boundary condition at the location of the initial interface. As described in [8], such a configuration is very rich in terms of the possible instability mechanisms observed when varying the two non-dimensional control parameters (see Sect. 2). Here we restrict our attention to the aforementioned delayed-double diffusive instability (DDD) and to several of its features that can be described only when studying it through non-modal analysis. In Sect. 2, we introduce the model used in our problem and briefly recall the fundamental ingredients of NMA. Section 3 is devoted to the analysis of some sample results focusing on the optimal time of introduction of the perturbation and the time evolution of the optimal wavenumber of the perturbation. Some concluding remarks are provided in Sect. 4.

2 Theoretical Framework

2.1 Model

We consider a miscible solution of solute A at initial concentration A_0 overlying another miscible solution of solute B at initial concentration B_0 in a vertically oriented 2-D porous medium. The horizontal axis is the y-axis, while the x-axis is the vertical axis pointing in the direction of gravity, with the miscible interface located at $x = 0$. The domain is assumed to be infinite in both directions. The flow is incompressible and, as we consider a porous medium, the governing equation for the velocity field is Darcy's law. The time evolution of concentrations A and B follows advection-diffusion equations. In their non-dimensional form, the governing equations then read [8],

$$\nabla p = -\mathbf{u} + (A + RB)\,\mathbf{i}_x, \;\; \nabla.\mathbf{u} = 0, \tag{1}$$

$$\frac{\partial A}{\partial t} + \mathbf{u}.\nabla A = \nabla^2 A, \;\; \frac{\partial B}{\partial t} + \mathbf{u}.\nabla B = \delta\,\nabla^2 B, \tag{2}$$

where $\mathbf{i}_x$ is the unit vector in the direction of gravity and the density ρ is equal to $A + RB$. The ratio of the diffusion coefficients is denoted $\delta = D_B/D_A$ and R is the buoyancy ratio that compares the density of the two solutions and that is defined as $R = \alpha_B B_0/\alpha_A A_0$, with α_A and α_B the solutal expansion coefficients.

2.2 Temporal Evolution of Perturbations

The previous equations are linearised by decomposing A and B into base states and perturbations periodic in the y-direction:

$$A = \bar{A}(x,t) + \epsilon\, a(x,t)\, \mathrm{e}^{iky}, \quad B = \bar{B}(x,t) + \epsilon\, b(x,t)\, \mathrm{e}^{iky}, \tag{3}$$

with $\epsilon \ll 1$ the initial amplitude of the perturbations. As we assume that the domain is infinite in the x-direction, the boundary conditions for the perturbations are $a(\pm\infty, t) = b(\pm\infty, t) = 0$. Base states obey diffusion equations and their analytical form is thus

$$\bar{A}(x,t) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right), \quad \bar{B}(x,t) = \frac{1}{2}\mathrm{erfc}\left(-\frac{x}{2\sqrt{\delta t}}\right). \tag{4}$$

The linearised equations are discretised by evaluating perturbations at each point x_i of a grid in the x-direction and spatial derivatives are computed with second-order central finite difference schemes. After linearisation and discretisation, the evolution of the perturbations can be computed using the operator T_k defined as:

$$\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t} = \mathsf{T}_k(t)\mathbf{c}, \quad \mathbf{c} = (a_0, a_1, \ldots, a_N, b_0, b_1, \ldots, b_N)^T, \tag{5}$$

with $a_i = a(x_i, t)$ and $b_i = b(x_i, t)$. T_k depends on time through the time-dependence of base states and is also non-normal (the subscript k denotes the wavenumber of the perturbation). To obtain optimal perturbations, we introduce the operator that transforms any initial perturbation into the solution of (5). It is the propagator $\mathsf{X}(t; t_0)$, defined through $\mathbf{c}(t) = \mathsf{X}_k(t; t_0)\mathbf{c}(t_0)$. Injecting this definition in (5) leads to the following matrix differential equation that we solve using a 4-order Runge–Kutta algorithm:

$$\frac{\mathrm{d}\mathsf{X}_k(t;t_0)}{\mathrm{d}t} = \mathsf{T}_k(t)\mathsf{X}_k(t;t_0), \quad \mathsf{X}_k(t_0;t_0) = \mathsf{I}. \tag{6}$$

2.3 Non-modal Analysis (NMA)

In order to perform NMA, we define the amplification $\Phi_k(t)$ at time t for a given wavenumber k as the ratio of the norm of a perturbation at time t and its initial norm, maximised over all initial perturbations (the norm is chosen as the 2-norm). By definition of the propagator and the matrix norm, $\Phi_k(t)$ then corresponds to the norm of the propagator:

$$\Phi_k(t) = \max_{\mathbf{c}_0} \frac{\|\mathbf{c}(t)\|_2}{\|\mathbf{c}(t_0)\|_2} = \max_{\mathbf{c}_0} \frac{\|\mathsf{X}_k(t;t_0)\mathbf{c}(t_0)\|_2}{\|\mathbf{c}(t_0)\|_2} = \|\mathsf{X}_k(t;t_0)\|_2. \tag{7}$$

One can show that the 2-norm of a matrix equals its largest singular value [2]. As a consequence, by doing the singular value decomposition of the propagator, which is an extension of the eigendecomposition of a matrix, we get the initial perturbation that will be the most amplified at time t, i.e. the optimal perturbation. We also define the maximum amplification $\Phi_{\max}(t)$ as the amplification $\Phi_k(t)$ maximised over all possible wavenumbers, $\Phi_{\max}(t) = \max_k \Phi_k(t)$. The initial perturbation corresponding to $\Phi_{\max}(t)$ is the perturbation that is optimal for a given set of parameters R and δ.

3 Delayed-Double Diffusion—Results

As recalled in the introduction, several types of instability mechanisms exist for the model considered depending on the values of R and δ. Here we focus on the so-called delayed-double diffusive instability at $R = 4$ and $\delta = 3$. This means that the lower solution is denser than the upper one, but contains a solute that diffuses faster than the one of the upper solution. This configuration is similar to the one encountered in the double diffusion instability but is qualitatively different since it is, in the QSSA modal sense, stable at $t_0 = 0$ and only becomes unstable at $t_0 \approx 2000$ (whereas the DD instability is already unstable at $t_0 = 0$). The density profile computed from the base states (4) is plotted in Fig. 1 as a function of time.

In time dependent systems, an important parameter for the stability analysis is the initial time t_0 at which perturbations are introduced. In Fig. 1 we plot the time evolution of $\Phi_{\max}$ for 5 different values of t_0. These times are the same as those chosen by Trevelyan et al. to perform the QSSA analysis of the DDD instability. From the point of view of NMA, we conclude that to reach an amplification of 10^2, the optimal time is around 10^4: an optimal perturbation introduced at 10^4 reaches this amplification faster than the perturbations introduced at 3000 and 5000. Note

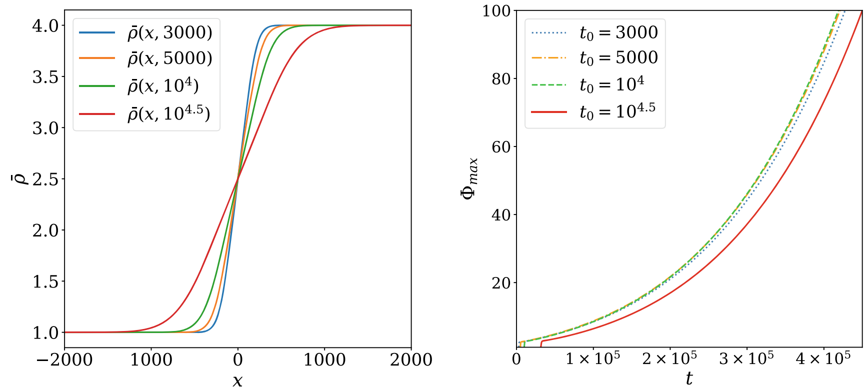

Fig. 1 **Left**: Base-state density profile $\bar{\rho} = \bar{A} + R\bar{B}$ at times $t = 3000, 5000, 10^4$ and $10^{4.5}$. **Right**: Maximum amplification computed with initial times $t = 3000, 5000, 10^4$ and $10^{4.5}$

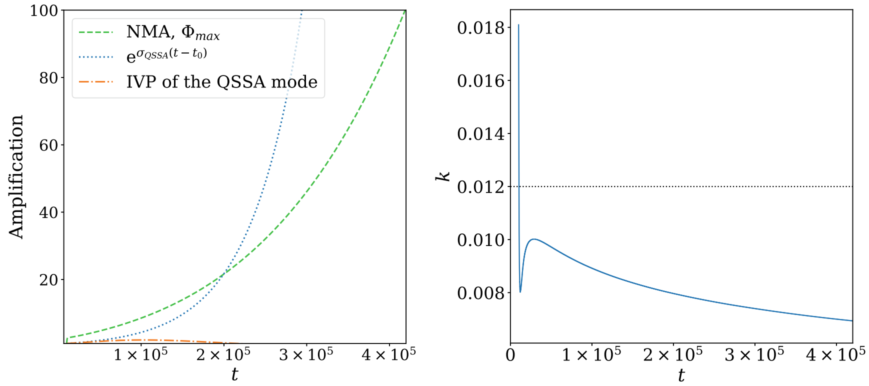

Fig. 2 **Left**: Amplification of the fastest growing mode computed by QSSA at initial time $t_0 = 10^4$ compared to optimal amplification and extrapolated exponential amplification. **Right**: Optimal wavenumber computed with initial time $t_0 = 10^4$

that the choice of amplification 10^2 is arbitrary. Only non-linear simulations can provide pertinent information concerning the necessary amplification required for the non-linear transition given an initial perturbation amplitude.

Interestingly, the maximum growth rate given by QSSA is also realized around $t_0 = 10^4$ and corresponds to a wavenumber $k = 0.012$ [8]. Based on this, QSSA predictions about the instability of the system such as the wavelength of the observed fingers and onset times are based on this value of k and the dynamics of the corresponding eigenvector. In order to test the relevance of this result, we have performed a time marching of this eigenvector with the linearized equations (5) and measured its amplification with time. This amplification is plotted in Fig. 2 (left) along with the maximum amplification $\Phi_{\max}$ and the extrapolated exponential amplification predicted by QSSA. We first observe that the extrapolated exponential amplification is much faster than the maximum amplification $\Phi_{\max}$ when it equals values greater than 20. Therefore σ_{QSSA} cannot serve as a reliable growth rate for the DDD case considered here beyond moderate amplifications. Second, we note that the eigenmode computed by QSSA does not undergo a significant amplification when it is used as an initial condition for an IVP and that it is in fact a stable mode of the time evolving system.

Finally, we examine in Fig. 2 (Right) the wavenumber of the optimal perturbation as a function of time for a perturbation introduced at $t_0 = 10^4$. After a sharp drop and subsequent rise, the value of k slowly decreases with t. On the same plot we have indicated the value of the wavenumber ($k = 0.012$) of the fastest growing QSSA mode. We conclude here that this QSSA mode is not a good indicator of the wavenumber of the optimal perturbation needed to achieve a significant amplification (here 10^2).

4 Conclusions and Future Work

The non-modal stability analysis (NMA) has been applied to the case of a delayed-double diffusion instability in porous media. By examining the maximum amplification and the corresponding optimal wavenumber, it is shown that for the delayed-double diffusion case, NMA predictions defer substantially from those of the linear analysis based on the quasi-steady-state approximation (QSSA). In a future work, the space of parameters R and δ will be further explored, and special attention will be paid to regions that are asymptotically stable according to QSSA. Indeed, because of the non-normality of T, perturbations could be amplified even if all QSSA modes decay. This transient growth could lead to significant amplification.

References

1. D. Daniel, N. Tilton, A. Riaz, Optimal perturbations of gravitationally unstable, transient boundary layers in porous media. J. Fluid Mech. **727**, 456–487 (2013)
2. R.A. Horn, C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 2013)
3. S. Rapaka, S. Chen, R.J. Pawar, P.H. Stauffer, D. Zhang, Non-modal growth of perturbations in density-driven convection in porous media. J. Fluid Mech. **609**, 285–303 (2008)
4. P.J. Schmid, D.S. Henningson, *Stability and Transition in Shear Flows*. Applied Mathematical Sciences, vol. 142 (Springer, New York, 2001)
5. A.C. Slim, T.S. Ramakrishnan, Onset and cessation of time-dependent, dissolution-driven convection in porous media. Phys. Fluids **22**, 124103 (2010)
6. M.E. Stern, The salt-fountain and thermohaline convection. Tellus **12**, 172–175 (1960)
7. C.T. Tan, G.M. Homsy, Stability of miscible displacements in porous media: rectilinear flow. Phys. Fluids **29**, 3549–3556 (1986)
8. P.M.J. Trevelyan, C. Almarcha, A. De Wit, Buoyancy-driven instabilities of miscible two-layer stratifications in porous media and Hele-Shaw cells. J. Fluid Mech. **670**, 38–65 (2011)