Encyclopedia of Nonlinear Science

Alwyn Scott
Editor

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its associated shift operator S^t is expressed through a one-dimensional map $f : z_n \mapsto z_{n+1}$, defined implicitly by $z_{n+1} - z_n = h(z_n + z_{n+1})$. This allows one to find conditions for IT or SIT for particular functions h (Sharkovsky, 1994; Sharkovsky et al., 1995). When the conditions of (6) are replaced with $w(0,t) = 0, \quad w_x(1,t) = h(w_x(0,t))$, there arises a two-dimensional map defined by $z_{n+1} - z_{n-1} = h(z_n)$.

There are many other one- and many-dimensional BVPs whose dynamics are described in terms of low-dimensional maps, as in the above examples. In these cases, the theory of maps suggests why and how turbulence occurs in the BVP and presents scenarios for self-structuring and self-stochastization. Of importance here are the following properties of maps: the intricate dynamical structure of the basins of attracting cycles, the local self-similarity of the set of points with unstable trajectories, and the occurrence of a smooth invariant measure. Figure 2 is an example of how processes of self-structuring lead to stochastic turbulence.

A.N. SHARKOVSKY AND E.YU. ROMANENKO

See also **Attractors; Butterfly effect; Chaotic dynamics; Dimensions; Dynamical systems; Ergodic theory; Maps; Measures; Mixing; One-dimensional maps; Phase space; Routes to chaos; Sinai–Ruelle–Bowen measures; Turbulence**

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TURING PATTERNS

As noted by D'Arcy Wentworth Thompson (1917) people in the early 1900s (when the study of symmetry-breaking instabilities was still in its infancy) had already considered the possibility of generating stationary regular concentration patterns through the interplay of diffusion and chemistry. At mid-century, the British mathematician and computing pioneer, Alan Turing, was the first to formulate necessary conditions for the occurrence of space symmetry breaking in the context of biological morphogenesis (Turing, 1952). Following the emergence of the self-oscillating Belousov–Zhabotinsky reaction (Epstein & Pojman, 1998) in the mid-1960s, Ilya Prigogine and coworkers revived Turing's concept, put it on sound thermodynamic and kinetic grounds, and showed that it could only be sustained in continuously fed reactors at a finite distance from equilibrium (Nicolis & Prigogine, 1977).

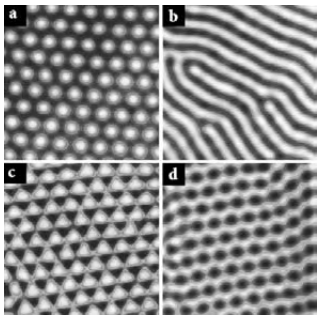

Figure 1. (a)–(d) Turing structures of different symmetries obtained with the chlorite-iodide-malonic acid reaction. Dark and light regions, respectively, correspond to high and low iodide concentration. The wavelength, a function of kinetic parameters and diffusion coefficients, is of the order of 0.2 mm. All patterns are at the same scale: view size 1.7mm × 1.7mm (Courtesy P. De Kepper, CRPP).

This work opened up a whole new field of physical chemistry. Many theoretical studies followed, and the diffusive instability that generates such dissipative structures has popped up in other domains of physics and chemistry (Ball, 1999). However, experiments in the chemical realm lagged behind, and it was only in 1989 that the first experimental evidence was obtained by De Kepper and his group (Castets et al., 1990) using the chlorite-iodide-malonic acid reactive system in so-called gel reactors. A recent detailed status of Turing patterns and other symmetry-breaking instabilities in solution chemistry is presented in Borckmans et al. (2002).

The Turing–Prigogine mechanism consists in the spontaneous instability of a homogeneous mixture of chemically reacting species, when some parameter threshold is crossed as one moves away from equilibrium conditions. It leads to stationary, space-periodic patterns for the concentrations of reactants (see Figure 1). In its minimal form, the description of all the systems that exhibit such diffusive instability can formally be cast in the common language of reaction-diffusion systems governed by the set of equations

$$\frac{\partial \boldsymbol{c}(\boldsymbol{r},t)}{\partial t} = \boldsymbol{f}(\boldsymbol{c},b) + \nabla \cdot D\nabla \boldsymbol{c}(\boldsymbol{r},t), \qquad (1)$$

where $\boldsymbol{c}(\boldsymbol{r}, t) \equiv (\ldots, c_i, \ldots)$ is the local concentration vector, $\boldsymbol{f}(\boldsymbol{c}, b)$ is a vector function representing the reaction kinetics wherein lies the source of nonlinearity, b stands for a set of control parameters, and D is the matrix of diffusive transport coefficients. Appropriate initial and boundary conditions, in relation with the experimental setup are added to complete the mathematical formulation.

To support such symmetry-breaking instability, the chemical kinetics must involve some type of positive feedback loop controlled at least by an activator species that reinforces its own changes, the latter being counterbalanced by an inhibitory process. Spatial structures can form when the inhibitory effects are transported by diffusion over a larger space range than that of the activating mechanism. An intuitive picture may be obtained when a single activator (A) and inhibitor (H) are present. A autocatalytically promotes its own production and that of H, while the latter opposes the production of A. Consider such system in a nonequilibrium homogeneous steady state (hss) and quench it beyond the instability threshold. The hss then becomes very sensitive. A slight local fluctuation of the concentration of A will increase while it also spreads to the surroundings through diffusion. It will also start producing some H that, however, will diffuse away much faster from the point where the fluctuation occurred as $D_{\mathrm{H}} > D_{\mathrm{A}}$. H thereby hinders the propagation of A. A localized peak of activator surrounded by a barrier of H is thus created. In extended systems, such peaks tend to emerge everywhere, randomly distributed, and their interactions lead to the periodic concentration patterns. The beauty of Turing's idea lies in the counterintuitive organization role of diffusive processes when they compete with the proper autocatalytic chemistry, while diffusion still locally strives to erase any concentration of inhomogeneity.

Theoretical work uses nonlinear kinetic models for $\boldsymbol{f}(\boldsymbol{c}, b)$ with a limited number of chemical species, typically two or three (Brusselator, Oregonator, CDIMA, etc.). These models stand as a compromise between a minimum of chemical realism and mathematical tractability. For their part, the experimental kinetic schemes usually involve many species, often not fully determined (Epstein & Pojman, 1998).

Analytical work that relies heavily on bifurcation theory (Nicolis & Prigogine, 1977; Manneville, 1990) allows one to determine, through the solution of amplitude equations, which structures of given symmetry are stable for specific conditions (pattern selection). The calculated bifurcation diagrams help to organize the results obtained by straightforward numerical integration of the reaction-diffusion equations. Both types of information may be used to interpret the experimental results. This pattern selection problem was already on Turing's mind when he stated (Turing, 1952): "Most of an organism, most of the time, is developing from one pattern into another, rather than from homogeneity into a pattern. One would like to be able to follow this more general process mathematically also."

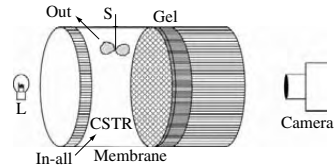

Figure 2. Schematic representation of a disc-shaped one side fed reactor (OSFR): CSTR (continuous stirred tank reactor), Membrane (mineral disc, pore size 0.02 mm) often placed to protect the gel from mechanical stress produced by the stirrer of the CSTR, Gel, In and Out (input and output ports of chemicals), L (light source), CCD camera.

The experimental work takes place in so-called open spatial reactors (Borckmans et al., 2002) which are specifically designed to control the reaction and the structures that eventually develop at a fixed distance from equilibrium and allow probing of the true asymptotic states of the reaction-diffusion systems. Experiments are now usually performed in a one-side fed reactor (OSFR) sketched in Figure 2. The core consists of a piece of soft hydrogel fed by diffusion through one of its faces with chemicals contained in a continuous stirred tank reactor (CSTR), the contents of which are continuously renewed by pumps. The other faces of the gel are pressed against impermeable transparent walls (Plexiglas). Viewing can be practiced both along the feeding axis or orthogonal to it (Ball, 1999; Ouyang & Swinney, 1991). The gel is used to avoid all perturbations induced by the hydrodynamic flows as those associated with the constant supply of fresh reactants, so that only reactive and diffusive processes compete. The necessary diffusion differential between activator and inhibitor species is obtained through the reversible binding of the activator molecules to the large molecular weight color indicator species that is included for visualization purposes. An advantage of such reactors is that they allow for direct correlations to be made between the dynamics of the CSTR, the bifurcation behaviors of which have been extensively studied in the past (Epstein & Pojman, 1998), and that of the gel.

Although scores of papers have been devoted to the application of Turing's idea to biological problems, this speculation remains to be confirmed (Epstein & Pojman, 1998; Borckmans et al., 2002).

PIERRE BORCKMANS AND GUY DEWEL

See also **Belousov–Zhabotinsky reaction; Brusselator; Morphogenesis, biological; Pattern formation; Reaction-diffusion systems**

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TWIST MAP

See **Nontwist maps**

TWISTOR THEORY

Introduced by Roger Penrose as a geometrical framework for the unification of quantum theory and general relativity (gravity), twistor theory brings out the complex (holomorphic) geometry that underlies real space-time. In general relativity, space-time is a four manifold with metric g. When $g = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$, where (t, x, y, z) are coordinates on $\mathbb{R}^4$, g is said to be flat with signature (1,3) and is called Minkowski space.

The first appearance of a complex structure arises from the fact that at a given event, the celestial sphere of light rays (null directions with respect to g) naturally has the structure of the Riemann sphere, $\mathbb{CP}^1$, in such a way that Lorentz transformations (linear transformations of the tangent space preserving the metric) act on this sphere by Möbius transformations. Twistor space extends this idea to the whole of Minkowski space. Denoted $\mathbb{PT}$, the twistor space for Minkowski space is complex projective three space, $\mathbb{CP}^3$, the space of one-dimensional subspaces of $\mathbb{C}^4$; it is a three-dimensional complex manifold obtained by adding a "plane at infinity" to $\mathbb{C}^3$. Physically, points of twistor space correspond to spinning massless particles in Minkowski space. Mathematically, the correspondence can be understood as the Klein correspondence.

The Klein Correspondence

The correspondence between $\mathbb{PT}$ and Minkowski space can be extended first to complexified Minkowski space, so that the coordinates are allowed to take on values in $\mathbb{C}$, and then to its conformal compactification by including some points at infinity. It then coincides with the classical complex Klein correspondence. The Klein correspondence is the one-to-one correspondence between lines in $\mathbb{CP}^3$ and points of a four complex-dimensional quadric, $\mathbb{CM}$, in $\mathbb{CP}^5$. The four-quadric $\mathbb{CM}$ can be understood as conformally compactified complexified Minkowski space. Introducing affine coordinates (λ, z_1, z_2) on $\mathbb{PT}$, we find that a line in $\mathbb{PT}$ corresponds to a point (t, x, y, z) by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} t - z & x + \mathrm{i}y \\ x - \mathrm{i}y & t + z \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

Alternatively, fixing (λ, z_1, z_2) in these equations gives a two-plane in complex Minkowski space corresponding to all the lines in $\mathbb{PT}$ through (λ, z_1, z_2). Such two-planes are called α-planes. They are totally null (i.e., the tangent vectors not only have zero length but are also mutually orthogonal) and also self-dual (under the differential geometer's notion of Hodge duality).

This complex correspondence can also be restricted to give correspondences for $\mathbb{R}^4$ with metrics of positive definite signature or ultra-hyperbolic (2, 2) signature.

The Penrose Transform

A basic task of twistor theory is to transform solutions to the field equations of mathematical physics into objects on twistor space. This works well for linear massless fields such as the Weyl neutrino equation, Maxwell's equations for electromagnetism, and linearized gravity. In its general form, this transform has become known as the Penrose transform. Such fields correspond to freely prescribable holomorphic functions $f(\lambda, z_1, z_2)$ (or, more precisely, analytic cohomology classes) on regions of twistor space. The field can be obtained from this function by means of a contour integral. The simplest of these integral formulae is

$$\phi(x^a) = \oint f\,(\lambda, t - z + \lambda(x + \mathrm{i}y), x - \mathrm{i}y + \lambda(t + z))\,\mathrm{d}\lambda, \tag{1}$$